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THE EFFECT OF COUPLE-STRESSES ON THE CORNER SINGULARITY DUE TO AN ASYMMETRIC SHEAR LOADING*

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Abstract—The plane-strain problem of an orthogonal elastic wedge, one face of which is subjected to arbitrary shearing tractions in the absence of other loads, is treated both within classical elastostatics and within a linear theory of elastic behavior that takes into account the influence of couple-stresses. If the loading fails to vanish at the apex, the conventional theory gives rise to corner singularities in the stress and rotation field, which stem from the incompatibility of the assumed boundary conditions with the symmetry of the stress tensor. These singularities are determined in closed elementary form and are found to agree with earlier results due to E. Reissner for the case of a uniform shear loading. It is shown next that the foregoing singularities disappear according to the couple-stress theory, in which the stress tensor is no longer required to be symmetric.

INTRODUCTION

THE linearized couple-stress theory of elastic behavior, explored by Mindlin and Tiersten [1], was applied in [2–4] to the solution of various singular plane-strain problems. The studies contained in [2–4] which are continued in the present paper, aim at the extent to which the pathological predictions of classical elastostatics in connection with singular stress-concentration problems are modified in the presence of couple-stresses.

The particular plane-strain problem to be considered here concerns a homogeneous and isotropic orthogonal elastic wedge (quarter-plane). One of the two wedge faces is subjected to an essentially arbitrary continuous distribution of shearing tractions, while all remaining surface tractions—as well as the body forces and body couples—are required to vanish identically.

In Section 1 we treat the foregoing problem within the classical theory of elasticity and use the Mellin transform to deduce an integral representation for the desired stress and rotation field. This integral representation is subsequently employed to examine the asymptotic behavior of the solution in the vicinity of the wedge corner. If the given shear loading fails to vanish at the corner, the assumed boundary conditions are incompatible with the prevailing symmetry of the stress tensor: in this instance the asymptotic analysis reveals the presence of a finite discontinuity in the stress field and a logarithmic infinity in the rotation field at the apex of the wedge. The detailed structure of these singularities is furnished in closed elementary form by the dominating terms in the asymptotic estimates obtained. As is to be anticipated on intuitive grounds, the leading terms referred to coincide with the elementary solution due to E. Reissner [5] for the case of a *uniformly* distributed shear loading.

Section 2 deals with the wedge problem under previous consideration within the framework of the linearized couple-stress theory. Thus the original boundary conditions are now

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supplemented by the requirement of vanishing couple-tractions on the wedge faces, while the classical field equations of plane strain are replaced by their counterpart in the modified theory. The modified quarter-plane problem is no longer amenable to a Mellin-transform technique. With the aid of results established in [2], however, it is reducible to a one-dimensional integral equation by superposition of the solutions to two half-plane problems that correspond to initially unknown distributions of shearing tractions. Further, this integral equation is shown to possess a solution the properties of which assure the absence of the corner singularities predicted by the classical theory: according to the modified theory all stresses and couple-stresses, as well as the rotation, remain finite and continuous in the closure of the quarter-plane, i.e. up to the apex of the wedge.

Finally, at the end of Section 2, we show that the analogue of Reissner's problem (uniform shear loading) in the couple-stress theory admits an elementary pseudo-solution, whose rotation and couple-stress field become unbounded at the wedge corner.

1. SOLUTION OF THE PROBLEM IN THE CLASSICAL THEORY. THE CLASSICAL CORNER SINGULARITY

Let (x_1, x_2) be two-dimensional rectangular cartesian coordinates and let D be the quarter-plane (see the diagram on the left in Fig. 1) defined by

$$D = \{(x_1, x_2)|0 < x_1 < \infty, 0 < x_2 < \infty\}. \tag{1.1}$$

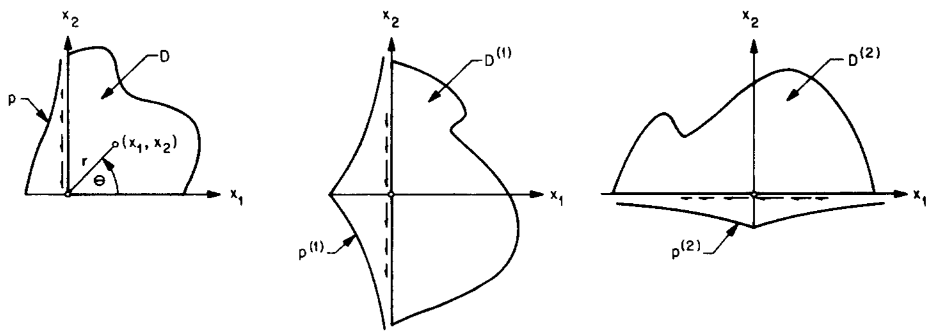

FIG. 1. Quarter-plane problem and auxiliary half-plane problems.

The classical plane-strain problem to be considered presently—referred to the cartesian coordinates (x_1, x_2) and cast in terms of Airy's stress function—may be formulated as follows:

We are to find an Airy function ϕ satisfying the biharmonic equation

$$\nabla^4\phi = 0 \tag{1.2}$$

on D, such that the stress field

$$\tau_{\alpha\beta} = \varepsilon_{\gamma\alpha}\varepsilon_{\rho\beta}\phi_{,\gamma\rho}, \tag{1.3}*$$

* Greek subscripts are understood to range over the integers (1, 2). Summation over repeated subscripts is implied and subscripts preceded by a comma indicate partial differentiation with respect to the corresponding cartesian coordinate. Observe that (1.3) imply the symmetry of the stress tensor.

in which $\varepsilon_{\gamma\alpha}$ denotes the components of the two-dimensional alternator, meets the boundary conditions

$$\left.\begin{aligned}\tau_{11}(0,x_2)&=0, \quad \tau_{12}(0,x_2)=p(x_2) \quad (0\le x_2<\infty),\\ \tau_{21}(x_1,0)&=0, \quad \tau_{22}(x_1,0)=0 \quad (0\le x_1<\infty),\end{aligned}\right\} \tag{1.4}$$

where p is a given load function, assumed to be continuously differentiable and absolutely integrable on $[0,\infty)$; in addition, the stresses (1.3) are to conform to the regularity conditions at infinity

$$\tau_{\alpha\beta}=o(1) \text{ as } x_\alpha x_\alpha\to\infty. \tag{1.5}$$

Once ϕ is known, the rotation field is obtainable by integration of the familiar relations

$$\omega_{,\alpha}=\frac{1-\nu}{2\mu}\varepsilon_{\beta\alpha}\nabla^2\phi_{,\beta}, \tag{1.6}$$

provided μ and ν designate the shear modulus and Poisson's ratio, respectively. Further, the associated displacements (in which we have no particular interest) may then be found by integrating the appropriate displacement–stress relations.

The foregoing boundary-value problem is most conveniently attacked in polar coordinates (r,θ) introduced through

$$x_1=r\cos\theta, \quad x_2=r\sin\theta \quad (0\le r<\infty, 0\le\theta<2\pi) \tag{1.7}$$

and by recourse to the Mellin transform. To this end we recall first that (1.2), (1.3), (1.4), (1.5), (1.6) in polar coordinates appear as

$$\nabla^4\phi\equiv\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]^2\phi=0 \quad \text{on } (0,\infty)\times(0,\pi/2), \tag{1.8*}$$

$$\tau_{rr}=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}, \quad \tau_{\theta\theta}=\frac{\partial^2\phi}{\partial r^2}, \quad \tau_{r\theta}=-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}, \tag{1.9}$$

$$\left.\begin{aligned}\tau_{\theta\theta}(r,\pi/2)&=0, \quad \tau_{r\theta}(r,\pi/2)=-p(r) \quad (0\le r<\infty),\\ \tau_{\theta\theta}(r,0)&=0, \quad \tau_{r\theta}(r,0)=0 \quad (0\le r<\infty),\end{aligned}\right\} \tag{1.10}$$

$$\tau_{rr}=o(1), \quad \tau_{\theta\theta}=o(1), \quad \tau_{r\theta}=o(1) \quad \text{as } r\to\infty, \tag{1.11}$$

$$\frac{\partial\omega}{\partial r}=-\frac{1-\nu}{2\mu r}\frac{\partial}{\partial\theta}\nabla^2\phi, \quad \frac{\partial\omega}{\partial\theta}=\frac{1-\nu}{2\mu}r\frac{\partial}{\partial r}\nabla^2\phi. \tag{1.12}$$

If f is a function defined and suitably regular on $[0,\infty)$, we denote the Mellin transform† of f by

$$\mathscr{T}\{f;s\}=\int_0^\infty f(r)r^{s-1}\,dr, \tag{1.13}$$

* Note that ϕ is now regarded as a function of (r,θ).

† See Doetsch [6] and Titchmarsh [7] for the theory of the Mellin transform. Formal expositions of the theory, with applications to elastostatics, are contained in the monographs by Tranter [8] and Sneddon [9]. A complete bibliography of the extensive literature on the use of the Mellin transform in connection with two-dimensional wedge problems in elasticity theory is beyond the scope of this paper; references up to 1958 may be found in [10].

s being the (complex) transform parameter. We now write $\hat{\phi}(s, \theta)$, $\hat{\tau}_{rr}(s, \theta)$, $\hat{\tau}_{\theta\theta}(s, \theta)$, $\hat{\tau}_{r\theta}(s, \theta)$, $\hat{\omega}(s, \theta)$, and $\hat{p}(s)$, in this order, for the Mellin transforms with respect to r of $\phi(r, \theta)$, $r^2\tau_{rr}(r, \theta)$, $r^2\tau_{\theta\theta}(r, \theta)$, $r^2\tau_{r\theta}(r, \theta)$, $r^2\omega(r, \theta)$, and $r^2p(r)$. Accordingly,

$$\left.\begin{aligned} \hat{\phi}(s, \theta) &= \int_0^\infty \phi(r, \theta) r^{s-1}\, dr, \qquad \hat{\tau}_{rr}(s, \theta) = \int_0^\infty \tau_{rr}(r, \theta) r^{s+1}\, dr \quad \text{etc.,} \\ \hat{\omega}(s, \theta) &= \int_0^\infty \omega(r, \theta) r^{s+1}\, dr, \qquad \hat{p}(s) = \int_0^\infty p(r) r^{s+1}\, dr. \end{aligned}\right\} \tag{1.14}$$

A formal application of the Mellin transform to (1.8), (1.9), (1.10), and (1.12) then yields*

$$\left[\frac{d^2}{d\theta^2} + s^2\right]\left[\frac{d^2}{d\theta^2} + (s+2)^2\right]\hat{\phi}(s, \theta) = 0 \qquad (0 < \theta < \pi/2), \tag{1.15}$$

$$\left.\begin{aligned} \hat{\tau}_{rr}(s, \theta) &= \left[\frac{d^2}{d\theta^2} - s\right]\hat{\phi}(s, \theta), \qquad \hat{\tau}_{\theta\theta}(s, \theta) = s(s+1)\hat{\phi}(s, \theta), \\ \hat{\tau}_{r\theta}(s, \theta) &= (s+1)\frac{d}{d\theta}\hat{\phi}(s, \theta), \end{aligned}\right\} \tag{1.16}$$

subject to the transformed boundary conditions

$$\hat{\tau}_{r\theta}(s, \pi/2) = -\hat{p}(s), \qquad \hat{\tau}_{\theta\theta}(s, \pi/2) = \hat{\tau}_{r\theta}(s, 0) = \hat{\tau}_{\theta\theta}(s, 0) = 0, \tag{1.17}$$

together with

$$\hat{\omega}(s, \theta) = \frac{1-\nu}{2\mu(s+2)}\left[\frac{d^3}{d\theta^3} + s^2\frac{d}{d\theta}\right]\hat{\phi}(s, \theta). \tag{1.18}$$

The boundary-value problem governed by (1.15), (1.16), (1.17) is elementary and (1.15) to (1.18) lead to the following results in the transform domain:

$$\left.\begin{aligned} \hat{\phi}(s, \theta) &= \frac{\hat{p}(s)}{2(s+1)\Delta(s)}[a_1(s, \theta) - a_2(s, \theta)\cos(2\theta) + a_3(s, \theta)\sin(2\theta)], \\ \hat{\tau}_{rr}(s, \theta) &= \frac{\hat{p}(s)}{2\Delta(s)}[-sa_1(s, \theta) + (s+4)a_2(s, \theta)\cos(2\theta) - (s+4)a_3(s, \theta)\sin(2\theta)], \\ \hat{\tau}_{\theta\theta}(s, \theta) &= \frac{\hat{p}(s)}{2\Delta(s)}[sa_1(s, \theta) - sa_2(s, \theta)\cos(2\theta) + sa_3(s, \theta)\sin(2\theta)], \\ \hat{\tau}_{r\theta}(s, \theta) &= \frac{\hat{p}(s)}{2\Delta(s)}[-sa_4(s, \theta) + (s+2)a_3(s, \theta)\cos(2\theta) + (s+2)a_2(s, \theta)\sin(2\theta)], \\ \hat{\omega}(s, \theta) &= -\frac{(1-\nu)\hat{p}(s)}{\mu\Delta(s)}[a_3(s, \theta)\cos(2\theta) + a_2(s, \theta)\sin(2\theta)], \end{aligned}\right\} \tag{1.19}$$

* See [8], art. 4.4 for details.

in which

$$\left.\begin{aligned} a_1(s,\theta) &= (s+1)\sin(s\pi/2)\cos(s\theta)-(s+2)\cos(s\pi/2)\sin(s\theta),\\ a_2(s,\theta) &= (s+1)\sin(s\pi/2)\cos(s\theta)-s\cos(s\pi/2)\sin(s\theta),\\ a_3(s,\theta) &= s\cos(s\pi/2)\cos(s\theta)+(s+1)\sin(s\pi/2)\sin(s\theta),\\ a_4(s,\theta) &= (s+2)\cos(s\pi/2)\cos(s\theta)+(s+1)\sin(s\pi/2)\sin(s\theta),\\ \Delta(s) &= (s+1)^2-\cos^2(s\pi/2). \end{aligned}\right\} \tag{1.20}$$

This completes the solution of the problem in the transform domain. We appeal next to the inversion theorem for the Mellin transform* and recall that (1.13) implies

$$f(r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathscr{T}\{f;s\}r^{-s}\,\mathrm{d}s \qquad (0<r<\infty) \tag{1.21}$$

for every choice of c such that $r^{c-1}f(r)$ is absolutely integrable on $[0,\infty)$, provided f is continuously differentiable on $[0,\infty)$. From (1.14), (1.21) one draws formally that

$$\left.\begin{aligned} \phi(r,\theta) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{\phi}(s,\theta)r^{-s}\,\mathrm{d}s,\\ \tau_{rr}(r,\theta) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{\tau}_{rr}(s,\theta)r^{-s-2}\,\mathrm{d}s \quad \text{etc.},\\ \omega(r,\theta) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{\omega}(s,\theta)r^{-s-2}\,\mathrm{d}s. \end{aligned}\right\} \tag{1.22}$$

The path of integration $\mathrm{Re}(s) = c$ in (1.22) must evidently lie within a common strip of regularity in the s-plane of the integrands in (1.22). This leads us to examine the behavior in the s-plane of $\hat{p}$ and of the functions defined by (1.19), (1.20).

Because of the assumed regularity of the load function p on $[0,\infty)$, the integral in the last of (1.14) is absolutely convergent for $-2 < \mathrm{Re}(s) \leq -1$. Hence† $\hat{p}(s)$ is analytic on the open strip $-2 < \mathrm{Re}(s) < -1$. Also, the regularity of p assures that

$$p(r) = p(0)+O(r) \quad \text{as } r\to 0, \qquad p(r) = O(r^{-1}) \quad \text{as } r\to\infty. \tag{1.23}$$

This enables one to prove‡ that $\hat{p}(s)$ may be continued analytically on to the strip $-3 < \mathrm{Re}(s) < -1$, except for a simple pole at $s = -2$ with the residue $p(0)$. Thus,

$$\hat{p}(s) = \frac{p(0)}{s+2}+g(s) \qquad (-3<\mathrm{Re}(s)<-1), \tag{1.24}$$

where g is a function analytic on $-3 < \mathrm{Re}(s) < -1$. Since all the functions defined in (1.20) are entire, it follows from (1.19) and the preceding conclusions regarding $\hat{p}(s)$ that $\hat{\phi}(s,\theta)$, $\hat{\tau}_{rr}(s,\theta)$, $\hat{\tau}_{\theta\theta}(s,\theta)$, $\hat{\tau}_{r\theta}(s,\theta)$, and $\hat{\omega}(s,\theta)$ are analytic on $-3 < \mathrm{Re}(s) < -1$ except possibly for poles that can occur only at the pole of $\hat{p}(s)$ and the zeros of $\Delta(s)$. Now $\Delta(s)$ vanishes on $-3 < \mathrm{Re}(s) < -1$ only at $s = -2$, where it has a simple zero. Consequently each of the integrands in (1.22) has at most one pole on $-3 < \mathrm{Re}(s) < -1$, which must be situated at

* See [7], p. 46, Theorem 28.
† See [6], volume 1, p. 59 and p. 144.
‡ See [6], volume 2, chapter 5, section 2 and chapter 4, section 2.

$s = -2$. Indeed, one finds that $\hat{\phi}(s,\theta)$, $\hat{\tau}_{rr}(s,\theta)$, $\hat{\tau}_{\theta\theta}(s,\theta)$, $\hat{\tau}_{r\theta}(s,\theta)$ all have simple poles at $s = -2$, whereas $\hat{\omega}(s,\theta)$ has a pole of the second order.

In view of the foregoing results we choose the path of integration for the inversion integrals (1.22) by taking

$$c = -1-\varepsilon \qquad (0 < \varepsilon < 1). \tag{1.25}$$

This choice of c insures that the path of integration lies within the common strip of regularity $-2 < \mathrm{Re}(s) < -1$ of all functions concerned and, by virtue of the inversion theorem cited earlier, guarantees that $p(r)$ is recovered from $\hat{p}(s)$ through

$$p(r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{p}(s) r^{-s-2}\,\mathrm{d}s \qquad (0 < r < \infty). \tag{1.26}$$

Moreover, (1.26) fails to hold true for $c < -2$ if $p(0) \neq 0$, as is apparent from (1.24). These observations motivate the choice of c made in (1.25).

It is not difficult to verify the formal solution given by (1.22), (1.25) together with (1.19), (1.20). Thus one can show that the integrals (1.22), with c determined by (1.25), represent real-valued functions, which possess continuous partial derivatives of all orders throughout the quarter-plane D; further, the functions ϕ, τ_{rr}, $\tau_{\theta\theta}$, $\tau_{r\theta}$, ω so defined satisfy the field equations (1.8), (1.9), (1.12) on D and obey the boundary conditions (1.10), as well as the regularity requirements at infinity (1.11). In the interest of brevity we omit the details of this *a posteriori* validation of the solution deduced earlier.

We turn now to an examination of the asymptotic behavior of the stress and rotation field in the vicinity of the wedge corner, which constitutes our main objective. To this end we consider first the stress τ_{rr} and note on the basis of the residue theorem that

$$\int_{\Gamma_1(b)} \hat{\tau}_{rr}(s,\theta) r^{-s-2}\,\mathrm{d}s = 2\pi i \mathring{\tau}_{rr}(r,\theta) - \sum_{k=2}^{4} \int_{\Gamma_k(b)} \hat{\tau}_{rr}(s,\theta) r^{-s-2}\,\mathrm{d}s, \tag{1.27}$$

where $\Gamma_k(b)$ $(k = 1, 2, 3, 4)$ are the four sides of the rectangular contour depicted in Fig. 2*, while $\mathring{\tau}_{rr}(r,\theta)$ is the residue of the integrand in (1.27) at $s = -2$. It is readily shown from

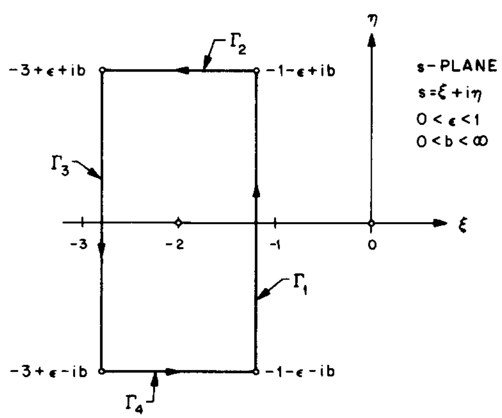

FIG. 2. Path of integration used in the asymptotic analysis of corner singularities.

* Note that Γ_k also depends on ε, which is however regarded as fixed.

(1.19) that the integrals over $\Gamma_2(b)$ and $\Gamma_4(b)$ in (1.27) tend to zero as $b \to \infty$ and that the integral over $\Gamma_3(b)$ is $o(1)$ as $r \to 0$. Consequently (1.22), (1.27) yield

$$\tau_{rr}(r,\theta) = \frac{1}{2\pi i} \lim_{b\to\infty} \int_{\Gamma_1(b)} \hat{\tau}_{rr}(s,\theta) r^{-s-2}\, ds = \mathring{\tau}_{rr}(r,\theta) + o(1) \qquad \text{as } r \to 0. \tag{1.28}$$

Proceeding in a strictly analogous manner in connection with $\tau_{\theta\theta}$, $\tau_{r\theta}$, and ω one arrives, upon computing the required residues at $s = -2$, at the following asymptotic estimates in the limit as $r \to 0$:

$$\left.\begin{aligned}
\tau_{rr}(r,\theta) &= \frac{p(0)}{2}\left\{2\theta + \sin(2\theta) - \frac{\pi}{2}[1+\cos(2\theta)]\right\} + o(1),\\
\tau_{\theta\theta}(r,\theta) &= \frac{p(0)}{2}\left\{2\theta - \sin(2\theta) - \frac{\pi}{2}[1-\cos(2\theta)]\right\} + o(1),\\
\tau_{r\theta}(r,\theta) &= \frac{p(0)}{2}\left[\frac{\pi}{2}\sin(2\theta) - 1 + \cos(2\theta)\right] + o(1),\\
\omega(r,\theta) &= -\frac{(1-\nu)}{\mu} p(0)\log r + o(1)
\end{aligned}\right\} \tag{129)*}$$

Equations (1.29) furnish for the cartesian components of stress and for ω, as $r \to 0$,

$$\left.\begin{aligned}
\tau_{11}(x_1,x_2) &= \frac{p(0)}{2}[-\pi + \sin(2\theta) + 2\theta] + o(1),\\
\tau_{22}(x_1,x_2) &= \frac{p(0)}{2}[-\sin(2\theta) + 2\theta] + o(1), \quad \tau_{12}(x_1,x_2) = p(0)\sin^2\theta + o(1),\\
\omega(x_1,x_2) &= -\frac{(1-\nu)}{\mu} p(0)\log r + o(1).
\end{aligned}\right\} \tag{1.30}$$

As is evident from (1.29), (1.30), if $p(0) \neq 0$, the stress field displays a finite discontinuity at the wedge corner, while the rotation field becomes logarithmically unbounded as $r \to 0$.

The leading terms in (1.30) by themselves satisfy the stress equations of equilibrium and compatibility on D; they give rise to a *uniform* shear loading of intensity $p(0)$ on the face $x_1 = 0, 0 \leq x_2 < \infty$ and otherwise to identically vanishing surface tractions. This particular solution of the field equations coincides with the singular solution exhibited by E. Reissner [5].

2. SOLUTION OF THE PROBLEM IN THE COUPLE-STRESS THEORY

We consider next the analogue in the couple-stress theory of plane strain† of the classical problem treated in Section 1. The field equations to be satisfied by the generalized Airy

* We omit in the formula for ω an unessential additive constant which corresponds to a rigid rotation of the entire body.

† See Mindlin [11] for an *ad-hoc* elementary discussion of the modified theory of plane strain. A more detailed discussion of this theory within the context of the three-dimensional couple-stress theory is contained in [2]. The two-dimensional theory is summarized in indicial notation in [3].

stress functions ϕ and ψ in cartesian coordinates take the form

$$2(1-\nu)l^2\varepsilon_{\alpha\beta}\nabla^2\phi_{,\beta} = (l^2\nabla^2\psi-\psi)_{,\alpha}, \tag{2.1}$$

provided l is the characteristic material length parameter, and (2.1) imply the uncoupled equations

$$\nabla^4\phi = 0, \qquad l^2\nabla^4\psi-\nabla^2\psi = 0. \tag{2.2}$$

We are at present required to construct a solution of (2.2) obeying (2.1) on the quarter-plane D, such that the ordinary stresses $\tau_{\alpha\beta}$ and the couple-stresses σ_α, derived from ϕ, ψ by means of

$$\tau_{\alpha\beta} = \varepsilon_{\gamma\alpha}\varepsilon_{\rho\beta}\phi_{,\gamma\rho}+\varepsilon_{\gamma\alpha}\psi_{,\beta\gamma}, \qquad \sigma_\alpha = \psi_{,\alpha}, \tag{2.3}$$

meet the boundary conditions

$$\left.\begin{aligned} &\tau_{11}(0,x_2) = 0, \quad \tau_{12}(0,x_2) = p(x_2), \quad \sigma_1(0,x_2) = 0 \quad (0 \le x_2 < \infty), \\ &\tau_{21}(x_1,0) = 0, \quad \tau_{22}(x_1,0) = 0, \quad \sigma_2(x_1,0) = 0 \quad (0 \le x_1 < \infty), \end{aligned}\right\} \tag{2.4}$$

and the regularity conditions at infinity

$$\tau_{\alpha\beta} = o(1), \qquad \sigma_\alpha = o(1) \qquad \text{as } x_\alpha x_\alpha \to \infty. \tag{2.5}$$

Here we assume the given load function p, together with its first two derivatives, to be continuous and absolutely integrable on $[0, \infty)$. Once ψ is known, the desired rotation field is obtainable by integration of the relations

$$4\mu l^2\omega_{,\alpha} = \psi_{,\alpha}. \tag{2.6}$$

The foregoing boundary-value problem—in contrast to its classical counterpart considered earlier—does not yield to a Mellin-transform technique because of the involvement of the Helmholtz operator in (2.1) and (2.2). The present quarter-plane problem may, however, be reduced to a one-dimensional integral equation with the aid of two auxiliary half-plane problems.*

Thus let $D^{(k)}$ $(k = 1, 2)$ be the two half-planes (see Fig. 1) defined by

$$\left.\begin{aligned} D^{(1)} &= \{(x_1,x_2)|0 < x_1 < \infty, \quad -\infty < x_2 < \infty\}, \\ D^{(2)} &= \{(x_1,x_2)|-\infty < x_1 < \infty, \quad 0 < x_2 < \infty\}, \end{aligned}\right\} \tag{2.7}$$

so that D is the intersection of $D^{(1)}$ and $D^{(2)}$. Further, let $\phi^{(k)}, \psi^{(k)}, \tau^{(k)}_{\alpha\beta}, \sigma^{(k)}_\alpha$, and $\omega^{(k)}$ $(k = 1, 2)$ satisfy the field equations (2.1), (2.3), (2.6) on $D^{(k)}$, meet the regularity conditions at infinity (2.5), as well as the respective boundary conditions

$$\tau^{(1)}_{11}(0,x_2) = 0, \quad \tau^{(1)}_{12}(0,x_2) = p^{(1)}(x_2), \quad \sigma^{(1)}_1(0,x_2) = 0 \quad (-\infty < x_2 < \infty), \tag{2.8}$$

$$\tau^{(2)}_{21}(x_1,0) = p^{(2)}(x_1), \quad \tau^{(2)}_{22}(x_1,0) = 0, \quad \sigma^{(2)}_2(x_1,0) = 0 \quad (-\infty < x_1 < \infty). \tag{2.9}$$

Accordingly each of the half-plane problems introduced above corresponds to shearing tractions along the bounding edge and otherwise to vanishing surface tractions. We restrict the load functions $p^{(k)}$ to be even, i.e.

$$p^{(k)}(-x) = p^{(k)}(x) \qquad (-\infty < x < \infty), \qquad (k = 1, 2). \tag{2.10}$$

* Cf. [12], Section 3.24, where the analogous reduction scheme is used in connection with a classical quarter-plane problem.

The solutions of the preceding auxiliary plane-strain problems for $D^{(1)}$ and $D^{(2)}$ were established in [2]* on the assumption that $p^{(k)}$ is absolutely integrable and piecewise smooth on $[0, \infty)$. We cite below from [2] the results obtained as far as the stress, couple-stress, and rotation field† is concerned.

On $D^{(1)}$:

$$\left.\begin{aligned}
\tau_{11}^{(1)}(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty b_{11}(x_1, s; l)\tilde{p}^{(1)}(s)\sin(x_2 s)\,ds,\\
\tau_{22}^{(1)}(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty b_{22}(x_1, s; l)\tilde{p}^{(1)}(s)\sin(x_2 s)\,ds,\\
\tau_{12}^{(1)}(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty b_{12}(x_1, s; l)\tilde{p}^{(1)}(s)\cos(x_2 s)\,ds,\\
\tau_{21}^{(1)}(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty b_{21}(x_1, s; l)\tilde{p}^{(1)}(s)\cos(x_2 s)\,ds,\\
\sigma_1^{(1)}(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty b_1(x_1, s; l)\tilde{p}^{(1)}(s)\cos(x_2 s)\,ds,\\
\sigma_2^{(1)}(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty b_2(x_1, s; l)\tilde{p}^{(1)}(s)\sin(x_2 s)\,ds,\\
\omega^{(1)}(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty b(x_1, s; l)\tilde{p}^{(1)}(s)\cos(x_2 s)\,ds.
\end{aligned}\right\} \qquad (2.11)‡$$

On $D^{(2)}$:

$$\left.\begin{aligned}
\tau_{11}^{(2)}(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty b_{22}(x_2, s; l)\tilde{p}^{(2)}(s)\sin(x_1 s)\,ds,\\
\tau_{22}^{(2)}(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty b_{11}(x_2, s; l)\tilde{p}^{(2)}(s)\sin(x_1 s)\,ds,\\
\tau_{12}^{(2)}(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty b_{21}(x_2, s; l)\tilde{p}^{(2)}(s)\cos(x_1 s)\,ds,\\
\tau_{21}^{(2)}(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty b_{12}(x_2, s; l)\tilde{p}^{(2)}(s)\cos(x_1 s)\,ds,\\
\sigma_1^{(2)}(x_1, x_2; l) &= -\frac{2}{\pi}\int_0^\infty b_2(x_2, s; l)\tilde{p}^{(2)}(s)\sin(x_1 s)\,ds,\\
\sigma_2^{(2)}(x_1, x_2; l) &= -\frac{2}{\pi}\int_0^\infty b_1(x_2, s; l)\tilde{p}^{(2)}(s)\cos(x_1 s)\,ds,\\
\omega^{(2)}(x_1, x_2; l) &= -\frac{2}{\pi}\int_0^\infty b(x_2, s; l)\tilde{p}^{(2)}(s)\cos(x_1 s)\,ds.
\end{aligned}\right\} \qquad (2.12)$$

* The present half-plane problems are in fact a special case of the problem solved in Section 3 of [2]. Note, however, that the coordinate frame associated with the second problem is obtained from the frame used in [2] by a rotation through $\pi/2$.

† The rotation field was not given explicitly in [2]; it is, however, immediately deducible from the results presented there.

‡ For the sake of clarity we make explicit from here on the dependence upon l of all functions considered. Thus we write $\tau_{\alpha\beta}^{(1)}(x_1, x_2; l)$ in place of $\tau_{\alpha\beta}^{(1)}(x_1, x_2)$, etc.

Here $\bar{p}^{(k)}$ designates the Fourier cosine-transform of $p^{(k)}$, given by

$$\bar{p}^{(k)}(s) = \int_0^\infty p^{(k)}(x)\cos(sx)\,dx \qquad (0 \le s < \infty), \qquad (k = 1, 2). \tag{2.13}$$

The auxiliary functions $b_{\alpha\beta}$, b_α, b appearing in (2.11) and (2.12) are accounted for through

$$\left.\begin{aligned}
b_{11}(x, s; l) &= \{xs\exp(-xs)+4(1-\nu)l^2s^2[\exp(-\alpha x/l)-\exp(-xs)]\}\frac{\alpha}{\beta},\\
b_{22}(x, s; l) &= \{(2-xs)\exp(-xs)-4(1-\nu)l^2s^2[\exp(-\alpha x/l)-\exp(-xs)]\}\frac{\alpha}{\beta},\\
b_{12}(x, s; l) &= \{\alpha(1-xs)\exp(-xs)-4(1-\nu)l^2s^2[ls\exp(-\alpha x/l)-\alpha\exp(-xs)]\}\frac{1}{\beta},\\
b_{21}(x, s; l) &= \{(1-xs)\exp(-xs)-4(1-\nu)ls[\alpha\exp(-\alpha x/l)-ls\exp(-xs)]\}\frac{\alpha}{\beta},\\
b_1(x, s; l) &= 4(1-\nu)l^2s[\exp(-\alpha x/l)-\exp(-xs)]\frac{\alpha}{\beta},\\
b_2(x, s; l) &= 4(1-\nu)l^2s[ls\exp(-\alpha x/l)-\alpha\exp(-xs)]\frac{1}{\beta},\\
b(x, s; l) &= -\frac{1-\nu}{\mu}[ls\exp(-\alpha x/l)-\alpha\exp(-xs)]\frac{1}{\beta},
\end{aligned}\right\} \tag{2.14}$$

in which

$$\alpha \equiv \alpha(ls) = (1+l^2s^2)^{\frac{1}{2}}, \qquad \beta \equiv \beta(ls) = \alpha(ls)+4(1-\nu)l^2s^2[\alpha(ls)-ls]. \tag{2.15}$$

Now set

$$\left.\begin{aligned}
\phi &= \phi^{(1)}+\phi^{(2)}, &\quad \tau_{\alpha\beta} &= \tau_{\alpha\beta}^{(1)}+\tau_{\alpha\beta}^{(2)},\\
\sigma_\alpha &= \sigma_\alpha^{(1)}+\sigma_\alpha^{(2)}, &\quad \omega &= \omega^{(1)}+\omega^{(2)} \quad \text{on } D
\end{aligned}\right\} \tag{2.16}$$

and note that the functions ϕ, $\tau_{\alpha\beta}$, σ_α, and ω so defined satisfy the field equations (2.1), (2.3), (2.6), as well as the regularity conditions at infinity (2.5), for *every* admissible choice of the load functions $p^{(1)}$ and $p^{(2)}$. We therefore seek to determine $p^{(k)}$ $(k = 1, 2)$ in such a way that $\tau_{\alpha\beta}$ and σ_α meet also the boundary conditions (2.4). In view of (2.8), (2.9), (2.11), (2.12), and (2.14), conditions (2.4) are fulfilled if and only if

$$p^{(1)}(x; l)+\frac{2}{\pi}\int_0^\infty b_{21}(x, s; l)\bar{p}^{(2)}(s; l)\,ds = p(x) \qquad (0 \le x < \infty), \tag{2.17}$$

$$\frac{2}{\pi}\int_0^\infty b_{21}(x, s; l)\bar{p}^{(1)}(s; l)\,ds+p^{(2)}(x; l) = 0 \qquad (0 \le x < \infty), \tag{2.18}$$

in which $\tilde{p}^{(k)}$ is the Fourier cosine-transform of $p^{(k)}$ introduced in (2.13). Applying the transform to (2.18) one obtains, on interchanging the order of integration*,

$$\tilde{p}^{(2)}(u;l) = -\int_0^\infty L(u,t;l)\tilde{p}^{(1)}(t;l)\,\mathrm{d}t \qquad (0 \le u < \infty), \tag{2.19}$$

where

$$L(u,t;l) = \frac{2}{\pi}\int_0^\infty \cos(ux)b_{21}(x,t;l)\,\mathrm{d}x. \tag{2.20}$$

Next, we substitute for $\tilde{p}^{(2)}$ from (2.19), (2.20) into (2.17), apply the cosine transform to the resulting equation and thus reach, after changing the order of the ensuing integrations,

$$\tilde{p}^{(1)}(u;l) - \int_0^\infty K(u,t;l)\tilde{p}^{(1)}(t;l)\,\mathrm{d}t = \tilde{p}(u) \qquad (0 \le u < \infty) \tag{2.21}$$

with

$$K(u,t;l) = \frac{4}{\pi^2}\int_0^\infty \mathrm{d}s \int_0^\infty \cos(ux)b_{21}(x,s;l)\,\mathrm{d}x \int_0^\infty \cos(sy)b_{21}(y,t;l)\,\mathrm{d}y. \tag{2.22}$$

As is apparent, the original quarter-plane problem under consideration has now been reduced to the one-dimensional integral equation (2.21): if $\tilde{p}^{(1)}$ satisfies (2.21) and $\tilde{p}^{(2)}$ is determined from (2.19), the solution of the problem is furnished by (2.16) with $\tau_{\alpha\beta}^{(k)}$, $\sigma_\alpha^{(k)}$, $\omega^{(k)}$ $(k = 1, 2)$ given by (2.11), (2.12) and (2.14), (2.15).

Equation (2.21) may, by an elementary change of variables, be transformed into a standard equation of Fredholm's second kind. For our particular purpose, however, it is more convenient to deal directly with (2.21). Our next objective is to show that for $l > 0$ this equation admits a unique solution $\tilde{p}^{(1)}$ which is absolutely integrable on $(0, \infty)$ and that the associated $\tilde{p}^{(2)}$ resulting from (2.19) has the same regularity property.† For this purpose we first obtain simplified representations for the kernels L and K presently given by (2.20) and (2.22), respectively. Since

$$\left.\begin{aligned} &\int_0^\infty \exp(-sx)\cos(ux)\,\mathrm{d}x = \frac{s}{s^2+u^2} \qquad (s > 0),\\ &\int_0^\infty x\exp(-sx)\cos(ux)\,\mathrm{d}x = \frac{s^2-u^2}{(s^2+u^2)^2} \qquad (s > 0), \end{aligned}\right\} \tag{2.23}$$

one draws from (2.20) and (2.14), (2.15) that

$$L(u,t;l) = \frac{4}{\pi}\frac{u^2 t}{(u^2+t^2)^2}\frac{\alpha(lt)}{\beta(lt)}\gamma(u,t;l) \tag{2.24}$$

for all (u, t) in $(0, \infty)\times(0, \infty)$, where

$$\gamma(u,t;l) = 1 - \frac{2(1-\nu)l^2(u^2+t^2)}{1+l^2(u^2+t^2)}, \tag{2.25}$$

* This formal manipulation, as well as subsequent reversals of iterated integrations, may be justified *a posteriori*

† This existence theorem is invalid if $l = 0$; the hypothesis $l > 0$ is taken for granted throughout the succeeding analysis.

while $\alpha(lt)$, $\beta(lt)$ are given by (2.15). Similarly, carrying out the integrations with respect to x and y in (2.22) with the aid of (2.23), one arrives at

$$K(u,t;l)=\frac{16}{\pi^2}\int_0^\infty \frac{u^2ts^3}{(u^2+s^2)^2(t^2+s^2)^2}\,\frac{\alpha(ls)\alpha(lt)}{\beta(ls)\beta(lt)}\gamma(u,s;l)\gamma(t,s;l)\,ds \tag{2.26}$$

for all (u,t) in $(0,\infty)\times(0,\infty)$.

Next we establish certain properties of the kernels L and K, which will be needed later on. It is easily confirmed from (2.15), (2.25) that

$$\left|\frac{\alpha(lt)}{\beta(lt)}\gamma(u,t;l)\right|<1 \qquad (0<u<\infty,\quad 0<t<\infty), \tag{2.27}$$

provided $0\le\nu\le\frac{1}{2}$. From (2.24), (2.27) follows

$$\int_0^\infty |L(u,t;l)|du<\frac{4}{\pi}\int_0^\infty\frac{u^2t\,du}{(u^2+t^2)^2}=1 \qquad (0<t<\infty). \tag{2.28}$$

Also, (2.26), (2.27) furnish

$$|K(u,t;l)|<\frac{16}{\pi^2}\int_0^\infty\frac{u^2ts^3\,ds}{(u^2+s^2)^2(t^2+s^2)^2} \qquad (0<u<\infty,\quad 0<t<\infty). \tag{2.29}$$

But

$$\frac{ts^3}{(t^2+s^2)^2}<1 \qquad (0<t<\infty,\quad 0<s<\infty), \tag{2.30}$$

and (2.29), (2.30) yield

$$|K(u,t;l)|<\frac{16}{\pi^2}\int_0^\infty\frac{u^2\,ds}{(u^2+s^2)^2}=\frac{4}{\pi u} \qquad (0<u<\infty,\quad 0<t<\infty). \tag{2.31}$$

Further, from (2.29) one has

$$\int_0^\infty |K(u,t;l)|\,du<\frac{16}{\pi^2}\int_0^\infty du\int_0^\infty\frac{u^2ts^3\,ds}{(u^2+s^2)^2(t^2+s^2)^2} \qquad (0<t<\infty). \tag{2.32}$$

The iterated integral in (2.32) can be evaluated by setting $u=r\cos\theta$, $s=r\sin\theta$ and is found to have the value $\pi^2/16$. Thus

$$\int_0^\infty |K(u,t;l)|\,du<1 \qquad (0<t<\infty). \tag{2.33}$$

Similarly we obtain

$$\int_0^\infty |K(u,t;l)|\,dt<\frac{4}{\pi^2} \qquad (0<u<\infty). \tag{2.34}$$

We now return to (2.21) and establish the following existence theorem: *The integral equation*

$$f(u)-\int_0^\infty K(u,t;l)f(t)\,dt=g(u) \qquad (0<u<\infty), \tag{2.35}$$

where

$$f(u) = \tilde{p}^{(1)}(u; l), \qquad g(u) = \tilde{p}(u) \qquad (0 < u < \infty), \tag{2.36}$$

for every fixed $l > 0$ *and every fixed* ν *in* $[0, \frac{1}{2}]$ *has one and only one solution that is absolutely integrable on* $(0, \infty)$.

To prove* this theorem we first conclude from (2.13), in view of the assumed regularity properties† of the load function p, that g is continuous on $[0, \infty)$ and, through integration by parts‡, that $g(u) = O(u^{-2})$ as $u \to \infty$. Therefore, g is absolutely integrable on $(0, \infty)$.

Let h be any function absolutely integrable on $(0, \infty)$. Then, by (2.33) and (2.34), after a permissible interchange in the order of the integrations,

$$\int_0^\infty du \int_0^\infty |K(u, t; l)|\,|h(t)|\,dt = \int_0^\infty |h(t)|\,dt \int_0^\infty |K(u, t; l)|\,du < \int_0^\infty |h(t)|\,dt, \tag{2.37}$$

whence there is a number ρ in $(0, 1)$, independent of h, such that

$$\int_0^\infty du \int_0^\infty |K(u, t; l)|\,|h(t)|\,dt \le \rho \int_0^\infty |h(t)|\,dt \tag{2.38}$$

Let $f^{(n)}$ $(n = 0, 1, 2, \ldots)$ be the sequence of functions defined by

$$f^{(0)}(u) = g(u), \qquad f^{(n+1)}(u) = g(u) + \int_0^\infty K(u, t; l) f^{(n)}(t)\,dt \qquad (0 < u < \infty). \tag{2.39}$$

It follows from (2.38), (2.39) by induction that

$$\int_0^\infty |f^{(n)}(u)|\,du \le \frac{1-\rho^{n+1}}{1-\rho} \int_0^\infty |g(u)|\,du, \tag{2.40}$$

whereas (2.38), (2.39), (2.40) and induction assure that

$$\int_0^\infty |f^{(n+k)}(u) - f^{(n)}(u)|\,du \le \rho^{n+1}\left(\frac{1-\rho^k}{1-\rho}\right) \int_0^\infty |g(u)|\,du. \tag{2.41}$$

Thus§ $f^{(n)}$ converges in the mean to a function f that is *absolutely integrable* on $(0, \infty)$.

We show further that $f^{(n)}$ converges to f pointwise on $(0, \infty)$. Indeed, by (2.39), (2.41) and (2.31),

$$|f^{(n+1+k)}(u) - f^{(n+1)}(u)| \le \frac{4}{\pi u} \rho^{n+1}\left(\frac{1-\rho^k}{1-\rho}\right) \int_0^\infty |g(t)|\,dt \qquad (0 < u < \infty), \tag{2.42}$$

so that

$$\lim_{n\to\infty} f^{(n)}(u) = f(u) \qquad (0 < u < \infty). \tag{2.43}$$

* The proof given below is similar to one given in Section 3.24 of [12] in connection with a closely related integral equation.

† See the assumptions on p immediately following (2.5).

‡ See also Erdélyi [13], p. 47.

§ See Titchmarsh [14], Sections 12.5 and 12.51.

To see that f satisfies (2.35), let $k \to \infty$ in (2.41). Taking this limit under the integral sign, as is legitimate*, we obtain

$$\int_0^\infty |f(u)-f^{(n)}(u)|\,du \le \frac{\rho^{n+1}}{1-\rho}\int_0^\infty |g(u)|\,du. \tag{2.44}$$

Now, from (2.39), (2.31) follows,

$$\begin{aligned} |f(u)-\int_0^\infty K(u,t;l)f(t)\,dt-g(u)| &= |f(u)-f^{(n)}(u)-\int_0^\infty K(u,t;l)[f(t)-f^{(n-1)}(t)]\,dt| \\ &\le |f(u)-f^{(n)}(u)|+\frac{4}{\pi u}\int_0^\infty |f(t)-f^{(n-1)}(t)|\,dt \qquad (0<u<\infty). \end{aligned} \tag{2.45}$$

But the right-hand member of (2.45) is arbitrarily small for sufficiently large n because of (2.43), (2.44), which verifies that *f conforms to the integral equation* (2.35). Finally, the *uniqueness* of the solution f is readily inferred from (2.35) with the aid of (2.33). This completes the proof.

That $\bar{p}^{(2)}$ obtained from (2.19) is also absolutely integrable follows immediately. For, from (2.19), (2.28) one draws

$$\int_0^\infty |\bar{p}^{(2)}(u;l)|\,du \le \int_0^\infty du\int_0^\infty |L(u,t;l)|\,|\bar{p}^{(1)}(t;l)|\,dt < \int_0^\infty |\bar{p}^{(1)}(t;l)|dt. \tag{2.46}$$

We are now in a position to turn to our main objective, which is to examine the asymptotic behavior in the vicinity of the wedge corner of the stress, couple-stress, and rotation fields given by (2.11) to (2.16). According to (2.14), (2.15), $b_{\alpha\pi}(x,s;l)$, $b_\alpha(x,s;l)$ and $b(x,s;l)$ are continuous and bounded functions for all (x,s) in $[0,\infty)\times[0,\infty)$ and $l>0$. Therefore, and since $\bar{p}^{(k)}$ $(k=1,2)$ determined by (2.19), (2.21) are now known to be absolutely integrable on $(0,\infty)$, the integrals in (2.11) and (2.12) define functions of (x_1,x_2) that are continuous† on $[0,\infty)\times[0,\infty)$. This fact enables us to conclude with the aid of (2.11) to (2.18) and the inversion theorem for the Fourier cosine-transform that as $r\to 0$, for fixed $l>0$,

$$\left.\begin{aligned} \tau_{\alpha\beta}(x_1,x_2;l) &= o(1)(\tau_{\alpha\beta}\ne\tau_{12}), \qquad \tau_{12}(x_1,x_2;l) = p(0)+o(1), \\ \sigma_\alpha(x_1,x_2;l) &= o(1), \qquad \omega(x_1,x_2;l) = o(1). \end{aligned}\right\} \tag{2.47‡}$$

Comparing (2.47) with (1.30), we note that *the discontinuities in $\tau_{\alpha\beta}$ and the logarithmic singularity in ω at the wedge corner predicted by the classical solution are no longer present when couple-stresses are taken into account.*

It may be instructive to point out where the argument leading to the conclusions (2.47), which are restricted to $l>0$, breaks down when $l=0$, i.e. in the classical theory. To this end we observe on the basis of (2.26), (2.15), (2.25) that

$$\int_0^\infty |K(u,t;0)|\,du = \int_0^\infty K(u,t;0)\,du = 1 \qquad (0<t<\infty). \tag{2.48}$$

Hence (2.38) now holds with $\rho=1$ and the remainder of the existence proof given earlier becomes invalid. In fact, we now show that the integral equation (2.21) with $l=0$ does not

* See Titchmarsh [14], Section 12.52.
† See, for example, Bromwich [15], Sections 171, 172.
‡ We again omit an additive constant in ω, which corresponds to a rigid rotation.

admit an absolutely integrable solution if $p(0) \neq 0$, i.e. if the given shear loading fails to vanish at the apex. Suppose, contrariwise, there exists $\tilde{p}^{(1)}$—absolutely integrable on $(0, \infty)$—such that

$$\tilde{p}^{(1)}(u;0) - \int_0^\infty K(u,t;0)\tilde{p}^{(1)}(t;0)\,dt = \tilde{p}(u) \qquad (0 \leq u < \infty). \tag{2.49}$$

Then, clearly,

$$\int_0^\infty \tilde{p}^{(1)}(u;0)\,du - \int_0^\infty \tilde{p}^{(1)}(t;0)\,dt \int_0^\infty K(u,t;0)\,du = \int_0^\infty \tilde{p}(u)\,du. \tag{2.50}$$

But (2.50), (2.48), in view of the inversion theorem for the Fourier-cosine transform*, imply

$$\int_0^\infty \tilde{p}(u)\,du \equiv \int_0^\infty du \int_0^\infty p(x)\cos(ux)\,dx = \frac{\pi}{2}p(0) = 0, \tag{2.51}$$

which contradicts the hypothesis $p(0) \neq 0$.

In the preceding analysis the load function p entering the boundary conditions (2.4) was required to vanish at infinity. Assume now $p(x_2)$ in (2.4) is replaced by a constant $p_0 \neq 0$, so that the applied shearing tractions are uniformly distributed for $0 \leq x_2 < \infty$. In this instance the regularity requirements (2.5) need to be relinquished. The problem thus arising is the counterpart in the modified theory of Reissner's [5] problem in the classical theory of plane strain. We presently employ Reissner's solution to generate an elementary pseudo-solution for the case of a uniform shear loading in the couple-stress theory.

For this purpose we recall from Section 1 that Reissner's results, as far as the stress and rotation fields are concerned, are given by the leading terms in (1.30) and thus take the form

$$\left.\begin{aligned} &\tau_{11}(x_1,x_2) = \frac{p_0}{2}[-\pi+\sin(2\theta)+2\theta],\\ &\tau_{22}(x_1,x_2) = \frac{p_0}{2}[-\sin(2\theta)+2\theta], \qquad \tau_{12}(x_1,x_2) = \tau_{21}(x_1,x_2) = p_0\sin^2\theta,\\ &\omega(x_1,x_2) = -\frac{(1-\nu)}{\mu}p_0\log r. \end{aligned}\right\} \tag{2.52}$$

Since ω in (2.52) is independent of θ, it is at once apparent that the normal derivative of this rotation field at the wedge boundary vanishes identically. It therefore follows from a theorem established in [3] (p. 74) that $\tau_{\alpha\beta}$ and ω of (2.52), supplemented by the couple-stress field

$$\sigma_\alpha = 4\mu l^2\omega_{,\alpha}, \tag{2.53}$$

conform to the field equations in the modified theory of plane strain and to the boundary conditions

$$\left.\begin{aligned} &\tau_{11}(0,x_2) = 0, \quad \tau_{12}(0,x_2) = p_0, \quad \sigma_1(0,x_2) = 0 \quad (0 \leq x_2 < \infty),\\ &\tau_{21}(x_1,0) = 0, \quad \tau_{22}(x_1,0) = 0, \quad \sigma_2(x_1,0) = 0 \quad (0 \leq x_1 < \infty). \end{aligned}\right\} \tag{2.54}$$

* See Titchmarsh [7], p. 13.

The explicit representation of σ_α furnished by (2.53) and the last of (2.52) is given by

$$\sigma_1(x_1, x_2) = -4(1-\nu)l^2 p_0 \frac{\cos\theta}{r}, \qquad \sigma_2(x_1, x_2) = -4(1-\nu)l^2 \frac{\sin\theta}{r}. \tag{2.55}$$

The solution (2.52), (2.55) (whose stress and rotation fields coincide with Reissner's solution) displays a discontinuity in the ordinary stress field and a logarithmic singularity in the rotation field at the apex, while its couple-stress field becomes unbounded as $r \to 0$ to the order $O(r^{-1})$. Since this singular behavior is inconsistent with the conclusions (2.47), according to which all fields under present consideration should be finite and continuous *up to* the wedge corner, it is clear that (2.52), (2.55) must be rejected as a physically irrelevant pseudo-solution to the analogue of Reissner's problem in the couple-stress theory. Pseudo-solutions of the same type to other singular problems in the modified theory were discussed in [3].

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Абстракт—Исследуется задача ортогонального упругого клина в перемещениях плоского состояния. Одна сторона этого клина загружена произвольными силами сдвига при отсуствии других нагрузок. Эта задача рассматривается в смысле классической статики упругого тела, как и в линейной теории упругости, которая учитывает влияние моментных напряжений. Если нагрузка изменяется так, чтобы она изчезла в вершине клина, конвенциональная теория дает рост угловых сингулярностей в напряжениях и поле ротации, которые происходят в виду несовместимости принятых граничных условий и симметрии тензора напряжений. Эти сингулярности, выражены в замкнутом, элементарном виде дают сходимость с более ранними результатами, выведенными Э.Рейсснером для случая постоянной нагрузки сдвигом. Далее показано, что сингулярности изчезают согласно теории моментных напряжений, в которой не требуется далее, чтобы тензор напражения являлся симметрическим.